ON NONSMOOTH SUBSPACES OF SPACES OF CONTINUITY FUNCTIONS

E.I. BEREZHNOI

In the geometric theory of Banach spaces of smooth functions the following problem is of considerable interest.

Question. *Suppose that we are give a Banach space Z of smooth functions. Does there exist an infinite-dimensional closed subspace $Y \subset Z$ such that each function $y \in Y$ not identically zero is not smoother than the nonsmoothest function from Z?*

This question was studied by many mathematicians. In the present report we give final answers of this problem in some situations.

Let $C[0,1]$ is Banach space of continuity function on $[0,1]$.

Suppose we are give a set $D \subset I = [0,1]$ of positive measure. For each $f : D \to R$, we define the modulus of continuity of f on D as follows. For each $h > 0$, we set $D_h = \{t \in D : t + h \in D\}$. Then by the modulus of continuity of the function f on D we mean

$$\Omega(f,h;D) = \sup_{0\le\delta\le h}\sup_{t\in D_\delta} |f(t+h) - f(t)|.$$

If $D = I$, then instead of $\Omega(f,h;I)$ we write $\Omega(f,h)$. Suppose that ω is a modulus of continuity. As is customary, by $H^\omega(D)$ we denote the function space

$$\|f|H^\omega(D)\| = \sup_{t\in D}|f(t)| + \sup_{h>0}\frac{\Omega(f,h;D)}{\omega(h)}.$$

Theorem 1. *There exists a closed infinite-dimensional subspace $E \subset C[0,1]$ such that each function $f \in E$ not identically zero has neither right- or left-hand finite derivative at any point $(0,1)$.*

Theorem 2. *Choose a Holder space H^ω. There exists a closed infinite-dimensional subspace $G \subset C[0,1]$, isomorphic to l^1 and such that, for each function $f \in G$ not identically zero, its restriction to any set of positive measure does not belong to $H^\omega(D)$.*

We denote $H^{0,\omega}$ the subset of space H^ω, such that for each function $f \in H^{0,\omega}$ exist $t_f \in (0,1)$, into hold one or two equality

$$\lim_{\delta\to+0}\sup_{h\le\delta}\frac{\Omega(f,h;D_\delta^+)}{\omega(h)} = 0, \qquad \lim_{\delta\to+0}\sup_{h\le\delta}\frac{\Omega(f,h;D_\delta^-)}{\omega(h)} = 0.$$

Here $D_\delta^+ = [t_f, t_f + \delta]$ and $D_\delta^- = [t_f - \delta, t_f]$.

Theorem 3. *Let ω is a modulus of continuity for with*

$$\lim_{h\to 0}\frac{\omega(h)}{h} = \infty, \tag{1}$$

and H^ω is Holder space There exists a closed infinite-dimensional subspace $G \subset H^\omega$, isomorphic to l^1 and such that, for each function $f \in G$ not identically zero, not belong to $H^{0,\omega}$.

SUPPORTED BY RFFI, N 08-01-00669

We define now Holder space with integral modulus of continuity.

Let X is symmetric space on $[0,1]$ and $\psi(X,t) = \|\chi(0,t)|X\|$ is fundamental function of space X. Symbol $\|f|X\|$ is a norm of element $f \in X$, $\chi(D_\delta)$ - is indicator function od set D_δ.

For each $f : D \to R$ we define the modulus of continuity of f on D into symmetric space as usual by formula

$$\Omega(f,h;D,X) = \sup_{0\le\delta\le h} \|(f(.+\delta) - f(.))\chi(D_\delta)|X\|.$$

If $D = I$, then instead of $\Omega(f,h;I,X)$ we write $\Omega(f,h;X)$. Suppose that ω is a modulus of continuity. As is customary, by $H^\omega(D)_X$ we denote the function space

$$\|f|H^\omega_X\| = \|f|X\| + \sup_{h\ge 0} \frac{\Omega(f,h;X)}{\omega(h)}.$$

By $H^{0,\omega}_X$ we denote the subset of space H^ω_X, such that for each function $f \in H^{0,\omega}_X$ exist $t_f \in (0,1)$, into hold one or two of equality

$$\lim_{\delta\to+0} \sup_{h\ge 0} \frac{\Omega(f,h;D^+_\delta,X)}{\psi(X,\delta)\omega(h)} = 0, \qquad \lim_{\delta\to+0} \sup_{h\ge 0} \frac{\Omega(f,h;D^-_\delta,X)}{\psi(X,\delta)\omega(h)} = 0.$$

The last theorem have in this case direct prototype.

Theorem 4. *Let ω is a modulus of continuity for with fulfil condition (1), X is symmetric space on $[0,1]$ and $\psi(X,t) = \|\chi(0,t)|X\|$ is fundamental function of space X. Let H^ω_X is Holder space.*

There exists a closed infinite-dimensional subspace $G \subset H^\omega_X$ such that each function $f \in G$ not identically zero not belong to $H^{0,\omega}_X$.

Let $\alpha \in (0,1), p \in [1,\infty)$. By Besov space $\Lambda^{\alpha,p}_X$ we denote the function space with norm

$$\|f|\Lambda^\alpha_{X,p}\| = (\sum_{i=1}^{\infty}(2^{\alpha i} \cdot \Omega(f,2^{-i};I,X))^p\big)^{1/p}.$$

Theorem 5. *Let X symmetric space on $[0,1]$, $\alpha \in (0,1), p \in [1,\infty)$. Let $\Lambda^{\alpha,p}_X$ is Besov space.*

There exists a closed infinite-dimensional subspace $G(\alpha_0,p,X)$ such that each function $f \in G(\alpha_0,p,X)$ not identically zero not belong to $\Lambda^{\alpha,p}_X$ for all $\alpha > \alpha_0$.

References

[1] Berezhnoi E. I. *The Subspace of $C[0,1]$ Consisting of Functions Having Finite One-Sided Derivatives Nowhere*, Mathematical Notes. **73**(3) (2003), 348-355

[2] Berezhnoi E. I. *A Subspace of Holder Space Consisting Only of Nonsmoothest Functions* Mathematical Notes, **74**(3) (2003), 329-340

[3] Berezhnoi E. I. *On Subspaces of Holder Spaces Consisting of Functions of Minimal Smoothness*, Doklady Mathematics. **70**(3) (2004), 911-913

[4] Berezhnoi E. I. *On Subspaces of $C[0,1]$ Consisting of Nonsmooth Functions*, Mathematical Notes, **81**(4) (2007), 13-16

Department of Mathematics, University of Yaroslavl, Yaroslavl, Sovjetskaya 14, Russia

E-mail address: `ber@uniyar.ac.ru`