SEMI-CONVEX SET MULTIFUNCTIONS

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In the last years, the non-additive and set-valued cases received a special attention because of their applications in mathematical economics, decision theory, computer and systems sciences, artificial intelligence, probabilities, statistics. Also, notions as (pseudo)atoms, Darboux property, non-atomicity, semi-convexity were intensively studied in recent years due to their applications in many classical measure theory problems and also in physics, convex analysis, theory of games.

In this paper we study semi-convexity of different types of set multifunctions defined on a ring $\mathcal{C}$ of subsets of an abstract nonvoid space T and taking values in the family of non-empty subsets of a real normed space X, $\mathcal{P}_0(X)$, endowed with the Hausdorff topology. We point out several relationships among semi-convexity, Darboux property and non-atomicity.

Let X be a real normed space, $\mathcal{P}_0(X)$ the family of all nonvoid subsets of X, $\mathcal{P}_{bf}(X)$ the family of all nonvoid closed bounded subsets of X, $\mathcal{P}_{bfc}(X)$ the family of all nonvoid closed bounded convex subsets of X, $\mathcal{P}_{kc}(X)$ the family of all nonvoid compact convex subsets of X and h the Hausdorff metric on $\mathcal{P}_{bf}(X)$.

Theorem 1. *I) If $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ has the Darboux property, then it is semi-convex.*

II) If $\mathcal{C}$ is a σ-ring and $\mu : \mathcal{C} \to \mathcal{P}_{bfc}(X)$ is a monotone semi-convex increasing convergent multimeasure, then μ has the Darboux property.

Example 1. I) Suppose $\mathcal{C}$ is the Borel σ-algebra of $T = [0, \infty)$ and $m : \mathcal{C} \to \mathbb{R}$ is a finitely additive set function so that m is increasing convergent and $m([t, s]) \leq s-t$, for every $t, s \in T$, with $t \leq s$ (for instance, m is the Lebesgue measure). Then the set multifunction $\mu : \mathcal{C} \to \mathcal{P}_0(\mathbb{R})$ defined by $\mu(A) = [0, m(A)]$, for every $A \in \mathcal{C}$, is semi-convex.

II) Let $\mathcal{C}$ be an algebra of subsets of T, X a Banach space, $\mu : \mathcal{C} \to \mathcal{P}_{kc}(X)$ a semi-convex multimeasure, $f : T \to \mathbb{R}$ a $\widetilde{\mu}$-totally-measurable bounded function on T and $M(A) = \int_A f d\mu$, for every $A \in \mathcal{C}$ (where $\int_A f d\mu$ is the Gould type set-valued integral introduced in [7]). Then M is also semi-convex.

Theorem 2. *Let $\mu : \mathcal{C} \to \mathcal{P}_{bf}(X)$ be monotone, null-additive and semi-convex. Then μ is non-atomic.*

In the sequel, T is a Hausdorff locally compact space and $\mathcal{C}$ is $\mathcal{B}_0$ (the Baire δ-ring generated by the G_δ-compact subsets of T that is, compact sets which are countable intersections of open sets) or $\mathcal{B}$ (the Borel δ-ring generated by the compact subsets of T).

Theorem 3. *If $\mu : \mathcal{C} \to \mathcal{P}_{bfc}(X)$ is a monotone decreasing convergent semi-convex multimeasure, then μ has the property:*

$(*)$ for every $t \in T$, there exists $A_t \in \mathcal{C}$ so that $t \in A_t$ and $\mu(A_t) = \{0\}$.

Corollary 2. *If $\mu : \mathcal{C} \to \mathcal{P}_{bfc}(X)$ is a $R'-$regular monotone multimeasure, then:*

Darboux property $\Rightarrow$ semi-convexity $\Rightarrow$ $(*)$ $\Rightarrow$ non-atomicity $\Leftrightarrow$ non-pseudo-atomicity.

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