

# ON THE NEGATIVE SPECTRUM OF ONE-DIMENSIONAL SCHRÖDINGER OPERATORS WITH POINT INTERACTIONS

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In the present talk we consider Schrödinger operators with  $\delta$ - and  $\delta'$ -interactions in  $L^2(\mathbb{R})$  which are formally defined by

$$(1) \quad L_{X,\alpha}f(x) = -\frac{d^2}{dx^2}f(x) + \sum_{k=1}^{\infty} \alpha_k \delta(x - x_k)f(x), \quad \alpha_k \in \mathbb{R},$$

and

$$(2) \quad L_{X,\beta}f(x) = -\frac{d^2}{dx^2}f(x) + \sum_{k=-\infty}^{\infty} \beta_k \delta'(x - x_k)f'_r(x_k), \quad \beta_k \in \mathbb{R},$$

where  $f'_r(x_k) = \frac{1}{2}[f'(x_k + 0) + f'(x_k - 0)]$  and (2) is understood in the sense of distributions.

Here  $X = \{x_k\}_{k \in I} \subset \mathbb{R}$  is an increasing sequence,  $d_k := x_{k+1} - x_k > 0$ ,  $k \in I$ , and  $I$  equals either  $\mathbb{N}$  or  $\mathbb{Z}$ . Moreover, we assume that the set  $X$  satisfies the following condition

$$(3) \quad d_* := \inf_{k \in I} d_k > 0, \quad d^* := \sup_{k \in I} d_k < \infty.$$

The operators associated with (1) and (2) we define  $L^2(\mathbb{R})$  by

$$(4) \quad L_{X,\alpha} = -\frac{d^2}{dx^2},$$

$$\text{dom}(L_{X,\alpha}) = \left\{ f \in W_2^2(\mathbb{R} \setminus X) : \begin{array}{l} f(x_k + 0) = f(x_k - 0), \\ f'(x_k + 0) - f'(x_k - 0) = \alpha_k f(x_k), \end{array} x_k \in X \right\},$$

$$(5) \quad L_{X,\beta} = -\frac{d^2}{dx^2},$$

$$\text{dom}(L_{X,\beta}) = \left\{ f \in W_2^2(\mathbb{R} \setminus X) : \begin{array}{l} f'(x_k + 0) = f'(x_k - 0), \\ f(x_k + 0) - f(x_k - 0) = \beta_k f'(x_k), \end{array} x_k \in X \right\}.$$

The operators  $L_{X,\alpha}$  and  $L_{X,\beta}$  are self-adjoint since the set  $X$  satisfies (3) (see [1] and also [7, 5]). We investigate the negative spectrum of operators with  $\delta$ - and  $\delta'$ -interactions (4) and (5) respectively.

During three last decades the theory of Schrödinger operators with point interactions have been attracting a lot of attention (numerous results and a comprehensive list of references may be found in [1, 2]). In the recent publications [3, 4], S. Albeverio and L. P. Nizhnik investigated the numbers  $\kappa_-(L_{X,\alpha})$  and  $\kappa_-(L_{X,\beta})$  of negative eigenvalues of the Schrödinger operators with a finite number of point interactions  $L_{X,\alpha}$  and  $L_{X,\beta}$ , respectively, i.e., in the case  $|X| = n < \infty$ . They obtained necessary and sufficient conditions for  $L_{X,\alpha}$  to have  $n$  negative squares in the case when all intensities are negative ([4, Theorem 2]).

It is also shown in [4, Theorem 6] that  $\kappa_-(L_{X,\beta})$  is maximal if and only if so is  $\kappa_-(\{\beta_k\}_{k=1}^n)$ , that is  $\kappa_-(L_{X,\beta}) = n$  if and only if  $\kappa_-(\{\beta_k\}_{k=1}^n) = n$ .

Using the concept of boundary triplets and the corresponding Weyl functions (see [9, 6]), we investigate the number of negative squares of the operators  $L_{X,\alpha}$  and  $L_{X,\beta}$  with both finite and infinite number of point interactions. This approach allows

us to complete and substantially generalize previous results from [3, 4] mentioned above.

Namely, we consider  $L_{X,\alpha}$  and  $L_{X,\beta}$  as self-adjoint extensions of the minimal symmetric operator

$$(6) \quad L_{\min} = -\frac{d^2}{dx^2}, \quad \text{dom}(L_{\min}) = \mathring{W}_2^2(\mathbb{R} \setminus X), \quad X = \{x_k\}_{k \in I}.$$

The adjoint operator  $L_{\min}^*$  of  $L_{\min}$  is determined by

$$(7) \quad L_{\min}^* = L_{\max} = -\frac{d^2}{dx^2}, \quad \text{dom } L_{\max} = W_2^2(\mathbb{R} \setminus X).$$

Using the boundary triplet  $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$  for  $L_{\min}^*$ , we describe spectral properties of self-adjoint extensions  $L_{X,\alpha}$  and  $L_{X,\beta}$  of  $L_{\min}$  by means of boundary conditions and the Weyl function corresponding to the boundary triplet  $\Pi$ . We obtain a description of negative squares of  $L_{X,\alpha}$  in terms of certain Jacobi matrices. This enables us to construct an algorithm for determining  $\kappa_-(L_{X,\alpha})$ .

Moreover, we provide a complete description of  $\kappa_-(L_{X,\beta})$ . Namely, for operators  $L_{X,\beta}$  with either finite or infinite number of  $\delta'$ -interactions we establish the equality  $\kappa_-(L_{X,\beta}) = \kappa_-(\beta)$ . In the case  $\kappa_-(\beta) = |X| = n < \infty$  this result implies [4, Theorem 6].

The results of the talk were obtained in collaboration with assistant professor L. Oridoroga (see <http://arxiv.org/abs/0903.1180>).

#### REFERENCES

- [1] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, H. Holden, *Solvable Models in Quantum Mechanics*, 2nd ed., AMS Chelsea Publ., 2004.
- [2] S. Albeverio, P. Kurasov, *Singular Perturbations of Differential Operators and Schrödinger Type Operators*, Cambridge Univ. Press, 2000.
- [3] S. Albeverio, L. Nizhnik, *On the Number of Negative Eigenvalues of One-Dimensional Schrödinger Operator with Point Interactions*, *Letters in Mathematical Physics*, **65** (2003), 27–35.
- [4] S. Albeverio, L. Nizhnik, *Schrödinger Operators with a number of Negative Eigenvalues equal to the number of Point Interactions*, *Methods Funct. Anal. Topology*, **9** (4) (2003), 273–286.
- [5] D. Buschmann, G. Stolz, J. Weidmann, *One-dimensional Schrödinger operators with local point interactions*, *J. Reine Angew. Math.*, 1995, 169–186.
- [6] V.A. Derkach, M.M. Malamud, *Generalised Resolvents and the boundary value problems for Hermitian Operators with gaps*, *J. Funct. Anal.*, **95** (1) (1991), 1–95.
- [7] F. Gesztesy, W. Kirsch, *One-dimensional Schrödinger operators with interactions singular on a discrete set*, *J. Reine Angew. Math.*, **362** (1985), 27–50.
- [8] N.I. Goloshchapova, L.L. Oridoroga, *The One-Dimensional Schrödinger Operator with Point  $\delta$ - and  $\delta'$ -interactions*, *Math. Notes*, **84** (1) (2008), 125–129.
- [9] V.I. Gorbachuk, M.L. Gorbachuk, *Boundary Value Problems for Operator Differential Equations*, *Mathematics and its Applications (Soviet Series)* 48, Kluwer Academic Publishers Group, Dordrecht, 1991.

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