ON COUNTABLY COMPACT BRANDT λ^0-EXTENSIONS OF TOPOLOGICAL SEMIGROUPS

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All topological spaces will be assumed to be Hausdorff. We shall follow the terminology of [1, 2, 3]. A topological space X is called *countably compact* if any countable open cover of X contains a finite subcover [3]. A *topological* (*inverse*) *semigroup* is a Hausdorff topological space with a continuous binary associative operation (and inversion).

Let S be a semigroup with zero and λ be a cardinal $\geqslant 1$. On the set $B_\lambda(S) = \lambda\times S\times\lambda\cup\{0\}$ we define the semigroup operation as follows

$$(\alpha,a,\beta)\cdot(\gamma,b,\delta)=\begin{cases}(\alpha,ab,\delta), & \text{if } \beta=\gamma;\\ 0, & \text{if } \beta\neq\gamma,\end{cases}$$

and $(\alpha,a,\beta)\cdot 0 = 0\cdot(\alpha,a,\beta)=0\cdot 0=0$, for all $\alpha,\beta,\gamma,\delta\in\lambda$ and $a,b\in S$. If $S=S^1$ then the semigroup $B_\lambda(S)$ is called the *Brandt λ-extension of the semigroup S* [4]. Obviously, $\mathcal{J}=\{0\}\cup\{(\alpha,\mathcal{O},\beta)\mid \mathcal{O}$ is the zero of $S\}$ is an ideal of $B_\lambda(S)$. We put $B^0_\lambda(S)=B_\lambda(S)/\mathcal{J}$ and we shall call $B^0_\lambda(S)$ the *Brandt λ^0-extension of the semigroup S with zero* [5].

Let $\mathcal{S}$ be some class of topological monoids with zero. Let λ be any cardinal $\geqslant 1$, and $(S,\tau)\in\mathcal{S}$. Let τ_B be a topology on $B^0_\lambda(S)$ such that $\big(B^0_\lambda(S),\tau_B\big)\in\mathcal{S}$ and $\tau_B|_{(\alpha,S,\alpha)}=\tau$ for some $\alpha\in\lambda$. Then $\big(B^0_\lambda(S),\tau_B\big)$ is called a *topological Brandt λ^0-extension of (S,τ) in $\mathcal{S}$*. If $\mathcal{S}$ coincides with the class of all topological monoids with zero, then $\big(B^0_\lambda(S),\tau_B\big)$ is called a *topological Brandt λ^0-extension of (S,τ)* [5].

A topological Brandt λ^0-extension $\big(B^0_\lambda(S),\tau_B\big)$ is called *compact* (resp., *countably compact*) if the topological space $\big(B^0_\lambda(S),\tau_B\big)$ is compact (resp., countably compact) [6]. Gutik and Repovš in [6] describe the structure of compact topological Brandt λ^0-extensions of topological monoids with zero.

Theorem 1. *A topological Brandt λ^0-extension $B^0_\lambda(S)$ of a topological monoid (S,τ) with zero is countably compact if and only if the cardinal $\lambda\geqslant 1$ is finite and (S,τ) is a countably compact topological semigroup. Moreover, for any countably compact topological monoid (S,τ) with zero and for any finite cardinal $\lambda\geqslant 1$ there exists an unique countably compact topological Brandt λ^0-extension $\big(B^0_\lambda(S),\tau_B\big)$ and the topology τ_B generated by the base $\mathcal{B}_B=\bigcup\{\mathcal{B}_B(t)\mid t\in B^0_\lambda(S)\}$, where: (i) $\mathcal{B}_B(t)=\{(\alpha,U(s)\setminus\{0_S\},\beta)\mid U(s)\in\mathcal{B}_S(s)\}$, when $t=(\alpha,s,\beta)$ is a non-zero element of $B^0_\lambda(S)$, $\alpha,\beta\in\lambda$ and $\mathcal{B}_S(s)$ is a base of the topology τ at $s\in S$; (ii) $\mathcal{B}_B(0)=\{\{0\}\cup\bigcup_{\alpha,\beta\in\lambda}(\alpha,U(0_S)\setminus\{0_S\},\beta)\mid U(0_S)\in\mathcal{B}_S(0_S)\}$, where 0 is zero of $B^0_\lambda(S)$, and $\mathcal{B}_S(s)$ is a base of the topology τ at the point $s\in S$.*

Theorem 2. *Every countably compact topological Brandt λ^0-extension $\big(B^0_\lambda(S),\tau_B\big)$ of a topological inverse semigroup (S,τ) is a topological inverse semigroup.*

Let S and T be topological monoids with zeros. Let $\texttt{CHom}_0(S,T)$ be a set of all continuous homomorphisms $\sigma\colon S\to T$ such that $(0_S)\sigma=0_T$. We put

$$\mathbf{E}^{top}_1(S,T)=\{e\in E(T)\mid \text{there exists } \sigma\in\texttt{CHom}_0(S,T) \text{ such that } (1_S)\sigma=e\}$$

and define the family $\mathcal{H}^{top}_1(S,T)=\{H(e)\mid e\in\mathbf{E}^{top}_1(S,T)\}$, where by $H(e)$ we denote the maximal subgroup with the unity e in the semigroup T. We shall say that a semigroup S has the *$\mathcal{B}^*$-property* if S does not contain the semigroup of 2×2-matrix units [6]. Also by $\mathfrak{CCTB}$ we denote the class of all countably compact topological monoids S with zero such that S has $\mathcal{B}^*$-property and every idempotent of S lies in the center of S.

We define a category $\mathcal{CCTB}_{\rm fin}$ as follows: $\mathbf{Ob}(\mathcal{CCTB}_{\rm fin})=\{(S,I)\mid S\in\mathfrak{CCTB}$ and I is a finite set$\}$, and if S is a trivial semigroup then we identify (S,I) and (S,J) for all finite sets I and J; and $\mathbf{Mor}(\mathcal{CCTB}_{\rm fin})$ consists of triples $(h,u,\varphi)\colon(S,I)\to(S',I')$, where: $h\colon S\to S'$ is a continuous homomorphism such that $h\in\mathtt{CHom}_0(S,S')$, $u\colon I\to H(e)$ is a map, for $H(e)\in\mathcal{H}_1^{top}(S,S')$, and $\varphi\colon I\to I'$ is an one-to-one map, with the composition $(h,u,\varphi)(h',u',\varphi')=(hh',[u,\varphi,h',u'],\varphi\varphi')$, where the map $[u,\varphi,h',u']\colon I\to H(e)$ is defined by the formula $(\alpha)[u,\varphi,h',u']=((\alpha)\varphi)u'\cdot((\alpha)u)h'$ for $\alpha\in I$.

We define a category $\mathcal{CCB}(\mathcal{TS})$ as follows. Let $\mathbf{Ob}(\mathcal{CCB}(\mathcal{TS}))$ be all countably compact topological Brandt λ^0-extensions of topological monoids S with zeros such that S has $\mathcal{B}^*$-property and every idempotent of S lies in the center of S. Let $\mathbf{Mor}(\mathcal{CCB}(\mathcal{TS}))$ be homomorphisms of countably compact topological Brandt λ^0-extensions of topological monoids S with zeros such that S has $\mathcal{B}^*$-property and every idempotent of S lies in the center of S.

For each $(S,I_{\lambda_1})\in\mathbf{Ob}(\mathcal{CCTB}_{\rm fin})$ with non-trivial S, let $\mathbf{B}(S,I_{\lambda_1})=B^0_{\lambda_1}(S)$ be the countably compact topological Brandt λ^0-extension of the topological monoid S. For each $(h,u,\varphi)\in\mathbf{Mor}(\mathcal{CCTB}_{\rm fin})$ with a non-trivial homomorphism h, where $(h,u,\varphi)\colon(S,I_{\lambda_1})\to(T,I_{\lambda_2})$ and $(T,I_{\lambda_2})\in\mathbf{Ob}(\mathcal{CCTB}_{\rm fin})$, we define a map $\mathbf{B}(h,u,\varphi)\colon \mathbf{B}(S,I_{\lambda_1})=B^0_{\lambda_1}(S)\to\mathbf{B}(T,I_{\lambda_2})=B^0_{\lambda_2}(T)$ as follows:

$$((\alpha,s,\beta))[\mathbf{B}(h,u,\varphi)]=\begin{cases}((\alpha)\varphi,(\alpha)u\cdot(s)h\cdot((\beta)u)^{-1},(\beta)\varphi), & \text{if}\quad s\notin S\setminus I_h;\\ 0_2, & \text{if}\quad s\in I_h^*,\end{cases}$$

and $(0_1)[\mathbf{B}(h,u,\varphi)]=0_2$, where $I_h=\{s\in S\mid (s)h=0_T\}$ is an ideal of S and 0_1 and 0_2 are the zeros of the semigroups $B^0_{\lambda_1}(S)$ and $B^0_{\lambda_2}(T)$, respectively. For each $(h,u,\varphi)\in\mathbf{Mor}(\mathcal{B})$ with a trivial homomorphism h we define a map $\mathbf{B}(h,u,\varphi)\colon \mathbf{B}(S,I_{\lambda_1})=B^0_{\lambda_1}(S)\to\mathbf{B}(T,I_{\lambda_2})=B^0_{\lambda_2}(T)$ as follows: $(a)[\mathbf{B}(h,u,\varphi)]=0_2$ for all $a\in\mathbf{B}(S,I_{\lambda_1})=B^0_{\lambda_1}(S)$. If S is a trivial semigroup then we define $\mathbf{B}(S,I_{\lambda_1})$ to be a trivial semigroup.

Theorem 3. *$\boldsymbol{B}$ is a full representative functor from $\mathcal{CCTB}_{\rm fin}$ into $\mathcal{CCB}(\mathcal{TS})$.*

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