

On closed graph theorem for Baire spaces

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A topological space X is called K -Souslin (respectively *quasi-Souslin*) if there is a map T from $\mathbb{N}^{\mathbb{N}}$ into the family $\mathcal{K}(X)$ of all compact subsets of X (into the family $\mathcal{P}(X)$ of all subsets of X) such that $\{T(\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ covers X and if $\alpha_n \rightarrow \alpha$ in $\mathbb{N}^{\mathbb{N}}$ and $x_n \in T(\alpha_n)$ for each $n \in \mathbb{N}$ then $\{x_n\}$ has a cluster point $x \in T(\alpha)$. A topological space X is called C -Souslin if there is a subspace Σ of $\mathbb{N}^{\mathbb{N}}$ and a map $T : \Sigma \rightarrow \mathcal{P}(X)$ such that $\bigcup \{T(\alpha) : \alpha \in \Sigma\} = X$ and if $\{\alpha_n\} \subseteq \Sigma$ converges in $\mathbb{N}^{\mathbb{N}}$ and $x_n \in T(\alpha_n)$ for every $n \in \mathbb{N}$ then $\{x_n\}$ has a cluster point in X . Every K -analytic space (in the sense of Choquet) is K -Souslin and every completely regular K -Souslin space is K -analytic. K -Souslin spaces are quasi-Souslin and quasi-Souslin spaces are C -Souslin.

Classic Schwartz's closed graph theorem for analytic spaces [7] was soon generalized by Martineau [5] and Nakamura [6], by showing that every linear mapping of closed graph from a Baire linear topological space into a K -Souslin linear topological space is continuous, then Valdivia [8] proved that every linear mapping of closed graph from a metrizable Baire linear topological space into a C -Souslin linear topological space is continuous and [9] that every linear mapping of closed graph from a locally convex hull of metrizable locally convex Baire spaces into a quasi-Souslin locally convex space is continuous. Very recently the first author, together with Kakol and López Pellicer, [3] has extended Valdivia's closed graph theorem for quasi-Souslin spaces. Other extensions of Martineau's theorem for topological groups can be found in [2, 8].

We present a closed graph theorem for Baire linear topological spaces in the domain and weakly C -Souslin linear topological spaces in the range which improves all aforementioned results. As application we extend an old result of De Wilde and Sunyach [1] and a recent one of Kakol and López Pellicer [4]. We also show that C -Souslin spaces are not productive.

References

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