MINIMAL LEFT IDEALS OF THE SUPEREXTENSIONS OF GROUPS

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In the talk we shall discuss the structure of minimal (left) ideals of the superextensions $\lambda(G)$ of Abelian groups G. By definition, a family $\mathcal{L}$ of subsets of a set X is called a *linked system* on X if $A \cap B$ is nonempty for all $A, B \in \mathcal{L}$. Such a linked system is maximal linked if it coincides with any linked system M on X that contains $\mathcal{L}$. The space $\lambda(X)$ of all maximal linked systems on X is called the *superextension* of X. It is endowed with the topology generated by the sub-base consisting of the sets $U^+ = \{\mathcal{L} \in \lambda(X) : U \in \mathcal{L}\}$, where U runs over subsets of X.

It is known [**?**], [**?**] that each binary operation $*$ on X extends to a right topological operation on $\beta(X)$, the Stone-Cech compactification of X, playing a crucial role in Combinatorics of Numbers. In the same way the operation $*$ can be further extended to a right-topological operation on $\lambda(X)$ by the formula:

$$\mathcal{U} * \mathcal{V} = \Big\{ \bigcup_{x \in U} x * V_x : U \in \mathcal{U},\ \{V_x\}_{x \in U} \subset \mathcal{V} \Big\}.$$

If the operation $*$ on X is associative, then it extends to an associative operation on $\lambda(X)$. In this case $\beta(X)$ is a subsemigroup of $\lambda(X)$, see [**?**].

The interest to studying the semigroup $\lambda(X)$ was motivated by the fact that for each maximal linked system $\mathcal{L}$ on X and each partition $X = A \cup B$ of X into two sets A, B either A or B belong to $\mathcal{L}$. This makes possible to apply maximal linked systems to combinatoral and Ramsey problems.

Understanding the structure of minimal left ideals of the semigroup $\beta(X)$ had important combinatorial consequences. For example, properties of ultrafilters from a minimal left ideal of $\beta(X)$ were exploited in the topological proof of the classical Van der Waerden Theorem. Minimal left ideals of the semigroup $\beta(\mathbb{Z})$ play also an important role in Topological Dynamics. We believe that studying the structure of minimal (left) ideals of the semigroups $\lambda(X)$ also will have some combinatorial or dynamical consequences.

We define a group G to be *odd* if the order of each element x of G is odd. If G is a finite odd group, then the maximal linked system

$$\mathcal{L} = \{A \subset G : |A| > |G|/2\}$$

is invariant in the sense that $x\mathcal{L} = \mathcal{L}$ for all $x \in G$. In fact, a group G possesses an invariant maximal linked system if and only if G is odd. A maximal linked system $\mathcal{Z} \in \lambda(G)$ on a group G is invariant if and only if $\mathcal{Z}$ is a right zero of the semigroup $\lambda(G)$ if and only if the singleton $\{\mathcal{Z}\}$ is a minimal left ideal in $\lambda(G)$. Taking into account that the invariant maximal linked systems form a closed right zeros subsemigroup of $\lambda(G)$, we obtain the following theorem:

Theorem 1. *A group G is odd if and only if all the minimal left ideals of $\lambda(G)$ are singletons. In this case the minimal ideal $K(\lambda(G))$ of $\lambda(G)$ is a closed right zeros semigroup consisting of invariant maximal linked systems.*

We recall that the group $\mathbb{Z}_2 = \varprojlim C_{2^k}$ of integer 2-adic numbers is a totally disconnected compact metrizable Abelian group, which is the limit of the inverse sequence

$$\cdots \to C_{2^n} \to \cdots \to C_8 \to C_4 \to C_2$$

of cyclic 2-groups C_{2^n}.

By the continuity of the functor λ in the category of compact Hausdorff spaces, the superextension $\lambda(\mathbb{Z}_2)$ can be identified with the limit of the inverse sequence

$$\cdots \to \lambda(C_{2^n}) \to \cdots \to \lambda(C_8) \to \lambda(C_4) \to \lambda(C_2)$$

of finite semigroups $\lambda(C_{2^k})$. This implies that $\lambda(\mathbb{Z}_2)$ is a metrizable zero-dimensional compact topological semigroup.

Theorem 2. *Minimal left ideals of the semigroup $\lambda(\mathbb{Z})$ are compact metrizable topological semigroups homeomorphic to minimal left ideals of the superextension $\lambda(\mathbb{Z}_2)$.*

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